Tutorial Notes 1

1. Evaluate
$$\int_R xy\cos y\,\mathrm{d}A,$$
where R: $-1 \le x \le 1, 0 \le y \le \pi$.

Solutions: By Fubini's theorem,
$$\int_R xy\cos y\,\mathrm{d}A = \int_0^\pi \left(\int_{-1}^1 xy\cos y\,\mathrm{d}x\right)\mathrm{d}y = 0.$$

2. Evaluate
$$\int_{-1}^1 \int_0^{\pi/2} x\sin\sqrt{y}\,\mathrm{d}y\,\mathrm{d}x.$$

Solutions: By Fubini's theorem,
$$\int_{-1}^1 \int_0^{\pi/2} x\sin\sqrt{y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/2} \left(\int_{-1}^1 x\sin\sqrt{y}\,\mathrm{d}x\right)\mathrm{d}y = 0.$$

3. Evaluate
$$\int_\Omega \mathrm{e}^s \log t\,\mathrm{d}A,$$
where Ω: $1 \le t \le 2, 0 \le s \le \log t$.

Solutions: By Fubini's theorem,
$$\begin{aligned}\int_\Omega \mathrm{e}^s \log t\,\mathrm{d}A &= \int_1^2 \left(\int_0^{\log t} \mathrm{e}^s \log t\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_1^2 (t-1)\log t\,\mathrm{d}t \\ &= \frac{(t-1)^2}{2}\log t\bigg|_1^2 - \int_1^2 \frac{(t-1)^2}{2}\cdot\frac{1}{t}\,\mathrm{d}t \\ &= \frac{\log 2}{2} - \int_1^2 \frac{t^2-2t+1}{2t}\,\mathrm{d}t \\ &= \frac{\log 2}{2} - \frac{1}{2}\left(\log 2 - \frac{1}{2}\right) \\ &= \frac{1}{4}.\end{aligned}$$

4. Evaluate
$$\int_D (y-2x^2)\,\mathrm{d}A,$$
where D: $|x|+|y| \le 1$.

Solutions: By symmetry,

$$\int_D y\,\mathrm{d}A = 0 \quad \text{and} \quad \int_D 2x^2\,\mathrm{d}A = 4\int_{D_{ru}} 2x^2\,\mathrm{d}A,$$

where D_{ru}: $x+y\le 1$, $x, y \ge 0$. By Fubini's theorem,

$$\int_{D_{ru}} 2x^2\,\mathrm{d}A = \int_0^1\left(\int_0^{1-x} 2x^2\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 2x^2(1-x)\,\mathrm{d}x = \frac{1}{6}.$$

Hence

$$\int_D (y-2x^2)\,\mathrm{d}A = -\frac{2}{3}.$$

5. Evaluate

$$\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,\mathrm{d}x.$$

Solutions: By the fundamental theorem of calculus and Fubini's theorem,

$$\begin{aligned}
&\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,\mathrm{d}x \\
=\ &\int_0^2 \tan^{-1} tx\Big|_{t=1}^{t=\pi}\,\mathrm{d}x \\
=\ &\int_0^2\left(\int_1^\pi \frac{x}{1+t^2x^2}\,\mathrm{d}t\right)\mathrm{d}x \\
=\ &\int_1^\pi\left(\int_0^2 \frac{x}{1+t^2x^2}\,\mathrm{d}x\right)\mathrm{d}t \\
=\ &\int_1^\pi \frac{1}{2t^2}\left(\int_0^{4t^2} \frac{1}{1+u}\,\mathrm{d}u\right)\mathrm{d}t \\
=\ &\int_1^\pi \frac{\log(1+4t^2)}{2t^2}\,\mathrm{d}t \\
=\ &-\frac{1}{2t}\log(1+4t^2)\Big|_1^\pi - \int_1^\pi\left(-\frac{1}{2t}\right)\frac{8t}{1+4t^2}\,\mathrm{d}t \\
=\ &\frac{\log 5}{2} - \frac{\log(1+4\pi^2)}{2\pi} + \int_1^\pi \frac{4}{1+4t^2}\,\mathrm{d}t \\
=\ &\frac{\log 5}{2} - \frac{\log(1+4\pi^2)}{2\pi} + \int_2^{2\pi} \frac{2}{1+t^2}\,\mathrm{d}t \\
=\ &\frac{\log 5}{2} - \frac{\log(1+4\pi^2)}{2\pi} + 2(\tan^{-1}2\pi - \tan^{-1}2).
\end{aligned}$$